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*Published in:*  
Quantum Science and Technology

*DOI:*  
[10.1088/2058-9565/aa6759](https://doi.org/10.1088/2058-9565/aa6759)

*Publication date:*  
2017

*Citation for published version (APA):*

Arenz, C., Burgarth, D., Giovannetti, V., Nakazato, H., & Yuasa, K. (2017). Lindbladian purification. *Quantum Science and Technology*, 2(2), [024001]. <https://doi.org/10.1088/2058-9565/aa6759>

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## Lindbladian purification

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To cite this article before publication: Arenz et al, 2017, Quantum Sci. Technol., at press:

<https://doi.org/10.1088/2058-9565/aa6759>

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# Lindbladian purification

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(Dated: March 17, 2017)

In a recent work [D. K. Burgarth *et al.*, Nat. Commun. **5**, 5173 (2014)] it was shown that a series of frequent measurements can project the dynamics of a quantum system onto a subspace in which the dynamics can be more complex. In this subspace even full controllability can be achieved, although the controllability over the system before the projection is very poor since the control Hamiltonians commute with each other. We can also think of the opposite: any Hamiltonians of a quantum system, which are in general noncommutative with each other, can be made commutative by embedding them in an extended Hilbert space, and thus the dynamics in the extended space becomes trivial and simple. This idea of making noncommutative Hamiltonians commutative is called “Hamiltonian purification.” The original noncommutative Hamiltonians are recovered by projecting the system back onto the original Hilbert space through frequent measurements. Here we generalize this idea to open-system dynamics by presenting a simple construction to make *Lindbladians*, as well as Hamiltonians, commutative on a larger space with an auxiliary system. We show that the original dynamics can be recovered through frequently measuring the auxiliary system in a non-selective way. Moreover, we provide a universal pair of Lindbladians which describes an “accessible” open quantum system for generic system sizes. This allows us to conclude that through a series of frequent non-selective measurements a nonaccessible open quantum system generally becomes accessible. This sheds further light on the role of measurement backaction on the control of quantum systems.

## I. INTRODUCTION

Noncommutativity is one of the key features of quantum mechanics. The order in which operations and/or measurements are performed influences the outcomes of an experiment. In particular in the Lie-theoretical approach to quantum control theory [1] the noncommutativity plays an important role. The goal of quantum control is to steer a quantum system to realize a desired transformation on it by shaping classical time-dependent fields [2]. Here the noncommutativity of the generators associated with the control fields influences the complexity of the resulting dynamics. For instance, for two commuting Hamiltonians  $H_1$  and  $H_2$ , which can be switched on and off by external control fields, the resulting unitary evolution is just equivalent to the one generated by a linear combination of  $H_1$  and  $H_2$ . On the contrary, by properly concatenating transformations induced by two noncommuting Hamiltonians one can produce effective evolutions associated with generators which are linearly independent of the original ones, enabling the system to explore more “directions.” For a finite-dimensional closed quantum system  $Q$ , the set of effective evolutions that can be implemented in this way is formed by the unitaries of the Lie group  $e^{\mathcal{L}(\mathbf{H})}$  generated by the dynamical Lie algebra  $\mathcal{L}(\mathbf{H})$  associated to the set of control Hamiltonians  $\mathbf{H} := \{H_1, H_2, \dots\}$ , i.e., the real vector space spanned by all possible linear combinations of the elements of  $\mathbf{H}$  and their iterated commutators [1, 3, 4]. Accordingly a close system  $Q$  characterized by a control set  $\mathbf{H}$  is said to be fully controllable if  $e^{\mathcal{L}(\mathbf{H})}$  includes all

possible unitary transformations on  $Q$ , or equivalently, if the dynamical Lie algebra  $\mathcal{L}(\mathbf{H})$  spans the whole operator algebra of  $Q$ , this last property of  $\mathbf{H}$  being also referred to as accessibility.

When it comes to open quantum systems, the characterization of the reachable (realizable) operations, as well as the associated notion of controllability, becomes more complicated since the allowed operations do not possess a group structure and the notion of dynamical generators is typically lost [5, 6]. A partial exception is provided by the subset of Markov processes which are equipped with a semigroup structure and admit the notion of dynamical generators, i.e., the super-operators of Gorini-Kossakowski-Lindblad-Sudarshan (GKLS) form [7, 8] (Lindbladians in the following). Still, also in this case, determining which dynamics can be activated by controlling a given collection  $\mathbf{L} := \{\mathcal{L}_1, \mathcal{L}_2, \dots\}$  of Lindbladians is a difficult unsolved problem. One would be tempted to tackle it by studying the Lie algebra  $\mathcal{L}(\mathbf{L})$  generated by  $\mathbf{L}$  and the corresponding Lie group  $e^{\mathcal{L}(\mathbf{L})}$ . However, at variance with the closed system scenario, linear combinations and commutators of elements of  $\mathbf{L}$  will in general produce super-operators which are no longer allowed dynamical generators (e.g., they cannot be cast in the GKLS form), or said it differently,  $e^{\mathcal{L}(\mathbf{L})}$  will include transformations which are unphysical. Furthermore, even for the elements of  $e^{\mathcal{L}(\mathbf{L})}$  which are physically allowed, it is in general not clear if it would be possible to implement them by simply playing with the control fields. In view of these facts for open quantum systems one distinguishes the physical notion of controllability, i.e., the ability of using  $\mathbf{L}$  and the classical fields which activate them to perform all physi-

cally allowed quantum transformations, from the weaker notion of accessibility, which in this case corresponds to have  $\mathfrak{L}(\mathbf{L})$  equal to whole Lie algebra generated by arbitrary Lindbladians. In the following, we will refer to this as the ‘GKLS algebra’ noting however that it contains many elements which are not of GKLS form. Differently from the closed quantum system case, it is indeed possible that a control set  $\mathbf{L}$  is accessible but not controllable. Still studying the accessibility of a collection of Lindbladians is a well posed mathematical problem, which can also shed light on the controllability issue, with accessibility being a necessary condition for controllability. Furthermore accessibility implies that the reachable set has non-zero volume and therefore has physical relevance: the short time dynamics explores a high dimensional space and is therefore of high complexity.

It turns out that almost all control sets  $\mathbf{L}$  are accessible [7]. Analogously to the case of closed systems, the key ingredient of this result can be identified with the noncommutativity of the elements of  $\mathbf{L}$ . But what about models where  $\mathbf{L}$  includes only mutually commuting Lindbladians? Is there a way to expand their algebra  $\mathfrak{L}(\mathbf{L})$  to cover the full GKLS algebra? For close quantum systems it has been observed that one can substantially change the dimension of the dynamical Lie algebra  $\mathfrak{L}(\mathbf{H})$  through frequently observing a part of the system [9], or by tampering it with a strong dissipative process that exhibits decoherence-free subspaces [10] (the gain being exponential in some cases). As a matter of fact, on the basis of the quantum Zeno effect [11], starting with a set of commuting control Hamiltonians  $\mathbf{H}$ , noncommutativity can be enforced through frequently projecting out part of the system onto a subspace where accessibility and hence full controllability is achieved. Also it has been observed that the projection trick can be reversed: specifically, starting from a set of noncommutative Hamiltonians  $\mathbf{H}$ , one can construct a new set  $\tilde{\mathbf{H}}$  formed by commutative elements on an extended Hilbert space which under projection reduces to the original one. This mechanism was studied in great detail in [12], where, borrowing from the notion of purification of mixed quantum states [13], the term *Hamiltonian purification* was introduced.

One may then ask whether a similar procedures can be applied to the algebra of a set  $\mathbf{L}$  of Lindbladians, namely, if it is possible to enlarge  $\mathfrak{L}(\mathbf{L})$  by means of some projection mechanisms and, on the contrary, if *Lindbladian purification* is always achievable. In this article we address these issues showing that indeed any set  $\mathbf{L}$  of Lindbladians can be “purified,” i.e., can be made commutative with each other, by embedding them in a larger space (note that the term “pure” was already used in [14] for Markovian generators in a slightly different way). To this end, we need to employ a different scheme from those for the Hamiltonian purification introduced in [12], since the naive application of the latter trivially violates some structural properties of GKLS generators (more details in the following). Our construction allows one to make Lindbladians and Hamiltonians

commutative on an extended space by means of an auxiliary system, which, through frequent non-selective measurements, yields the original noncommutative dynamics. Moreover, we present a universal pair of Lindbladians that generate the full GKLS dynamical Lie algebra for generic system sizes, the analysis providing us with a short and elementary proof of the generic accessibility [7]. Applying hence the Lindbladian purification procedure to such a universal set, we then show that almost all open systems become accessible, even though their generators are commutative with each other, by performing frequent non-selective measurements on a part of the system.

This article is organized as follows. Along the lines of [12] we begin in Sec. II by reviewing the definition of Hamiltonian purification and presenting an explicit construction for purifying an arbitrary number of Lindbladians and Hamiltonians. In Sec. III we consider the accessibility of controlled master equations. Concluding remarks are given in Sec. IV, and some details on the derivation of the projected dynamics and the proof of accessibility are provided in the Appendices.

## II. LINDBLADIAN PURIFICATION AND NON-SELECTIVE ZENO MEASUREMENTS

To begin with we first review the definition of Hamiltonian purification [12]. Suppose that we have  $n$  control Hamiltonians, which are switched on and off to steer a  $d$ -dimensional quantum system  $Q$ . Let  $\mathbf{H} = \{H_1, \dots, H_n\}$  be the set of the control Hamiltonians acting on the Hilbert space  $\mathcal{H}_d$  of  $Q$ , and  $\tilde{\mathbf{H}} = \{\tilde{H}_1, \dots, \tilde{H}_n\}$  be a corresponding set of Hamiltonians acting on an extended Hilbert space  $\mathcal{H}_{d_E}$  of dimension  $d_E (> d)$ , which includes  $\mathcal{H}_d$  as a proper subspace. We call  $\tilde{\mathbf{H}}$  a purifying set of  $\mathbf{H}$  if all the elements of  $\tilde{\mathbf{H}}$  commute with each other,

$$\tilde{H}_i \tilde{H}_j = \tilde{H}_j \tilde{H}_i, \quad \forall i, j \in \{1, \dots, n\}, \quad (1)$$

and they are related to those from  $\mathbf{H}$  through

$$H_j = P \tilde{H}_j P, \quad \forall j \in \{1, \dots, n\}, \quad (2)$$

with  $P$  being the projection onto  $\mathcal{H}_d$ . For a generic set  $\mathbf{H}$  consisting of  $n$  linearly independent Hamiltonians it can be shown [12] that there always exists an  $\tilde{\mathbf{H}}$  where the minimal dimension  $d_E^{(\min)}$  of the extended Hilbert space is bounded above by  $d_E^{(\min)} \leq nd$ . For instance for the case with  $n = 2$  Hamiltonians  $H_1$  and  $H_2$ , Proposition 1 of Ref. [12] states that a purifying set can be constructed on  $\mathcal{H}_{d_E} = \mathcal{H}_d \otimes \mathcal{H}_{d_A}$  with an auxiliary single qubit Hilbert space  $\mathcal{H}_{d_A}$ , the purifications and the projector being

$$\begin{aligned} \tilde{H}_1 &= H_1 \otimes \mathbb{1}_2 + H_2 \otimes \sigma_x, \\ \tilde{H}_2 &= H_2 \otimes \mathbb{1}_2 + H_1 \otimes \sigma_x, \end{aligned} \quad (3)$$

$$P = \mathbb{1}_d \otimes \frac{\mathbb{1}_2 + \sigma_z}{2}, \quad (4)$$

with  $\sigma_x$ ,  $\sigma_z$ , and  $\mathbb{1}_2$  the Pauli and the identity operators of the auxiliary qubit, respectively. The mapping  $\tilde{H}_j \rightarrow H_j$  can finally be realized through the quantum Zeno effect [11, 15] by frequently monitoring the extended system via a von Neumann measurement which projects the system onto  $\mathcal{H}_d$ , i.e.,

$$\lim_{N \rightarrow \infty} (P e^{-i\tilde{H}_j t/N} P)^N = e^{-iH_j t} P. \quad (5)$$

The question arises if an analogous construction can be extended to the case of Lindbladians. Specifically consider a set  $\mathbf{L} = \{\mathcal{L}_1, \dots, \mathcal{L}_n\}$  of  $n$  GKLS generators operating on a target system  $Q$ ,

$$\mathcal{L}_j = \mathcal{K}_j + \mathcal{D}_j, \quad j \in \{1, \dots, n\}, \quad (6)$$

with  $\mathcal{K}_j$  and  $\mathcal{D}_j$  being the Hamiltonian and dissipator contributions, i.e., the super-operators

$$\mathcal{K}_j(\dots) = -i[H_j, \dots], \quad (7)$$

$$\mathcal{D}_j(\dots) = \sum_{\alpha} [2L_{j,\alpha}(\dots)L_{j,\alpha}^\dagger - L_{j,\alpha}^\dagger L_{j,\alpha}(\dots) - (\dots)L_{j,\alpha}^\dagger L_{j,\alpha}], \quad (8)$$

$L_{j,\alpha}$  being the Lindblad operators acting on the Hilbert space  $\mathcal{H}_d$  of  $Q$ . We ask whether if it is possible to associate with  $\mathbf{L}$  a purifying set  $\tilde{\mathbf{L}} = \{\tilde{\mathcal{L}}_1, \dots, \tilde{\mathcal{L}}_n\}$  formed by GKLS generators possibly acting on an extended system, which are mutually commuting, i.e.,

$$\tilde{\mathcal{L}}_i \circ \tilde{\mathcal{L}}_j = \tilde{\mathcal{L}}_j \circ \tilde{\mathcal{L}}_i, \quad \forall i, j \in \{1, \dots, n\}, \quad (9)$$

from which one can recover the original elements via a projective mapping that should mimics (5) (in the above expressions we used the symbol “ $\circ$ ” to indicate the composition of super-operators).

A natural guess for identifying  $\tilde{\mathbf{L}}$  and the projective mapping would be to simply transporting the purification schemes of Ref. [12] at super-operator level, or equivalently, to represent the  $\mathcal{L}_j$ s as operators in Liouville space [16] and then simply applying to them the Hamiltonian purification scheme. This simple trick however does not work because, for instance, mapping as (3) will take positive operators into non-positive one, hence spoiling one fundamental property of GKLS generators. Another problem comes from the fact that for Markovian open systems described by Lindbladians, the quantum Zeno effect, which as we have seen is responsible for the implementation of the mapping  $\tilde{H}_j \rightarrow H_j$ , does not take place: a Markovian system can leak from one subspace specified by the projection operator belonging to a measurement outcome even in the limit of infinitely frequent projective measurements. In spite of these issues however a Lindbladian purification scheme can be obtained with the following simple construction:

*A purifying set  $\tilde{\mathbf{L}}$  can always be constructed by introducing an auxiliary Hilbert space  $\mathcal{H}_n$  of dimension  $n$  and*

*identifying the Hamiltonians  $\{\tilde{H}_j\}$  and the Lindblad operators  $\{\tilde{L}_{j,\alpha}\}$  of the purifying element  $\tilde{\mathcal{L}}_j = \tilde{\mathcal{K}}_j + \tilde{\mathcal{D}}_j$  as*

$$\begin{cases} \tilde{H}_j = nH_j \otimes |j\rangle\langle j|, \\ \tilde{L}_{j,\alpha} = \sqrt{n}L_{j,\alpha} \otimes |j\rangle\langle j|, \end{cases} \quad j \in \{1, \dots, n\}, \quad (10)$$

*with  $\{|j\rangle\}_{j=1}^n$  being an orthonormal basis for  $\mathcal{H}_n$ .*

Obviously through such a construction the operators  $\{\tilde{H}_j\}$  and  $\{\tilde{L}_{j,\alpha}\}$  commute with each other for different  $j$ , trivially ensuring the requirement (9). Regarding the analog of (5), we focus on non-selective projective measurement [17, 18] operating on the auxiliary system, i.e., the completely positive and trace preserving (CPTP) mapping of the form

$$\mathcal{P}(\dots) = \sum_k P_k(\dots)P_k, \quad (11)$$

given in terms of a complete set of orthonormal projection operators  $\{P_k\}$  corresponding to measurement outcomes and satisfying  $P_k P_{k'} = \delta_{kk'} P_k$  and  $\sum_k P_k = \mathbb{1}$ . Notice that if we perform  $(N+1)$  of such non-selective measurements at regular time intervals  $t/N$  during the evolution driven by a Lindbladian  $\mathcal{L}$ , the system will evolve according to the CPTP transformation

$$\begin{aligned} \Phi_{t,N}^{(\mathcal{L})} &:= (\mathcal{P} \circ e^{\mathcal{L}t/N} \circ \mathcal{P})^N \\ &= \left[ \text{id} + (\mathcal{P} \circ \mathcal{L} \circ \mathcal{P}) \frac{t}{N} + O\left(\frac{t^2}{N^2}\right) \right]^N \circ \mathcal{P}, \end{aligned} \quad (12)$$

which in the limit of  $N \rightarrow \infty$  converges to

$$\Phi_{t,\infty}^{(\mathcal{L})} = \lim_{N \rightarrow \infty} \Phi_{t,N}^{(\mathcal{L})} = e^{(\mathcal{P} \circ \mathcal{L} \circ \mathcal{P})t} \circ \mathcal{P}, \quad (13)$$

where  $\text{id}$  is the identity map and where we used the idempotent property  $\mathcal{P} = \mathcal{P} \circ \mathcal{P} = \mathcal{P}^2$  of (11). Equation (13) can also be derived following a perturbative approach with a strong amplitude-damping channel inducing the projection  $\mathcal{P}$  [19–21]. In our construction Eq. (13) is the formal counterpart of the Zeno limit (5): it shows that alternating the dynamics induced by a GKLS generator  $\mathcal{L}$  with  $\mathcal{P}$  induces on the system an evolution which can be effectively described in terms of an effective dynamical generator described by the projected super-operator  $\mathcal{P} \circ \mathcal{L} \circ \mathcal{P}$ . It should be stressed that the latter is not in GKLS form, i.e., it is not a Lindbladian. Indeed it acts as a proper Lindbladian only within the subspace specified by the super-projector  $\mathcal{P}$ , but the map  $e^{(\mathcal{P} \circ \mathcal{L} \circ \mathcal{P})t}$  itself is not CPTP (an explicit example of this fact is provided in Appendix A). Still we are going to identify (13) with the mechanism that yields the original Lindbladians  $\mathcal{L}_j \in \mathbf{L}$  expressed in the form (6)–(8) from their purified counterparts  $\tilde{\mathcal{L}}_j$  of with (10). For this purpose we assume the projectors  $P_k$  in (11) to be of the form

$$P_k = \mathbb{1} \otimes |\phi_k\rangle\langle\phi_k|, \quad (14)$$

where  $\{|\phi_k\rangle\}_{k=1}^{d_A}$  is an orthonormal basis for the auxiliary Hilbert space which is chosen to be mutually unbiased [22] against the orthonormal basis  $\{|j\rangle\}_{j=1}^{d_A}$  used for the purification (10). Then, as shown in Appendix B, one can verify that under the transformation (13) a generic density operator  $\rho_Q(0)$  for the original system, obtained by taking the trace over the auxiliary Hilbert space  $\mathcal{H}_{d_A}$ , evolves according to

$$\rho_Q(t) = e^{\mathcal{L}_j t} \rho_Q(0), \quad (15)$$

recovering hence the original dynamics generated by the unpurified Lindbladian  $\mathcal{L}_j$ .

As a simple example of Lindbladian purification we consider amplitude damping (AD) and pure dephasing (PD) in  $x$  direction of a single qubit. Within the Born-Markov approximation the corresponding Lindbladians  $\mathcal{L}_{AD}$  and  $\mathcal{L}_{PD}$  are typically used to describe the main noise sources in two level systems [13, 23]. The Lindblad operators read  $L_{AD} = \sigma$ ,  $L_{PD} = \sigma + \sigma^\dagger$  where  $\sigma = |0\rangle\langle 1|$  is the atomic lowering operator and we note that  $\mathcal{L}_{AD}$  and  $\mathcal{L}_{PD}$  do not commute. According to (10), a purifying set  $\{\tilde{\mathcal{L}}_{AD}, \tilde{\mathcal{L}}_{PD}\}$  is obtained through the purified Lindblad operators

$$\tilde{L}_{AD} = \sigma \otimes \sigma \sigma^\dagger, \quad (16)$$

$$\tilde{L}_{PD} = (\sigma + \sigma^\dagger) \otimes \sigma^\dagger \sigma, \quad (17)$$

and a frequent non-selective measurement (11) with projectors  $P_\pm = \mathbb{1} \otimes |\pm\rangle\langle \pm|$  where  $|\pm\rangle = 1/\sqrt{2}(|0\rangle \pm |1\rangle)$  recovers the unpurified dynamics. We note here that Lindblad operators similar to the purified versions (16) and (17) were introduced in [24] for bosonic systems.

### III. ACCESSIBILITY

We now turn our attention to the question on how frequent non-selective measurements can enrich the algebra  $\mathfrak{L}(\mathbf{L})$  of a Markovian open quantum system described by a collection  $\mathbf{L}$  of controlled generators. Specifically we shall focus on systems driven by master equations of the form

$$\frac{\partial}{\partial t} \rho(t) = \mathcal{L}(t) \rho(t), \quad (18)$$

where the super-operator  $\mathcal{L}(t) = \mathcal{K}(t) + \mathcal{D}$  is provided by a constant dissipative part represented by Lindblad operators  $L_\alpha$ , and by a time-dependent Hamiltonian term  $\mathcal{K}(t)(\cdots) = -i[H(t), \cdots]$  with

$$H(t) = H_0 + \sum_{k=1}^m u_k(t) H_k, \quad (19)$$

$\{u_k(t)\}_{k=1}^m$  being classical control fields that can be operated to switch on and off  $m$  control Hamiltonians  $\{H_k\}_{k=1}^m$ . This corresponds to having a control set

$\mathbf{L} := \{\mathcal{L}_0, \mathcal{K}_1, \dots, \mathcal{K}_m\}$  consisting of a drift (unmodulated) term

$$\mathcal{L}_0 = \mathcal{K}_0 + \mathcal{D}, \quad (20)$$

that includes both the dissipative part  $\mathcal{D}$  and the Hamiltonian contribution  $\mathcal{K}_0(\cdots) = -i[H_0, \cdots]$ , and of the set of Hamiltonian control generators

$$\mathcal{K}_k(\cdots) = -i[H_k, \cdots], \quad k \in \{1, \dots, m\}. \quad (21)$$

As already mentioned in the introduction, for a closed quantum system, i.e., without the dissipative part  $\mathcal{D}$ , the algebra  $\mathfrak{L}(\mathbf{L})$  associated with  $\mathbf{L}$  (i.e., the set of all real linear combinations and iterated commutators of these elements, drift term included) will fully characterize the set of unitary operations that can be implemented through shaping the control functions  $\{u_k(t)\}_{k=1}^m$ . For an open quantum system described by the master equation (18), instead,  $\mathfrak{L}(\mathbf{L})$  only characterizes the accessibility of the system. For a detailed analysis of the general structure of  $\mathfrak{L}(\mathbf{L})$  and simple examples, we refer to [7]. Here we focus instead on studying how the purification mechanism can influence the dimension of  $\mathfrak{L}(\mathbf{L})$ . In particular we shall see how a set of commutative Lindbladians can be turned into a new set of noncommutative Lindbladians which grant accessibility to the full GKLS algebra via the projection through frequent measurements.

To show this we start by showing that it is possible to identify a set  $\mathbf{L}$  formed by just a pair of Lindbladians whose algebra  $\mathfrak{L}(\mathbf{L})$  spans the full GKLS algebra. We therefore first prove that the pair

$$\mathcal{L}_0 = -i \text{ad}_{H_0} + \mathcal{D}_{|1\rangle\langle 2|}, \quad (22)$$

$$\mathcal{K} = -i \text{ad}_{|1\rangle\langle 1|} \quad (23)$$

with

$$H_0 = \sum_{j=1}^{d-1} |j\rangle\langle j+1| + \text{h.c.}, \quad (24)$$

where

$$-i \text{ad}_H(\cdots) = -i[H, \cdots], \quad (25)$$

$$\mathcal{D}_L(\cdots) = 2L(\cdots)L^\dagger - L^\dagger L(\cdots) - (\cdots)L^\dagger L, \quad (26)$$

does the job, namely, every possible Lindbladian can be generated by linear combinations and iterated commutators of (22) and (23). We only sketch the main steps here, whereas the details can be found in Appendix C. In the following we also use the notations

$$\mathcal{D}_{A,B}(\cdots) = 2B(\cdots)A^\dagger - A^\dagger B(\cdots) - (\cdots)A^\dagger B, \quad (27)$$

$$\text{Ad}_U(\cdots) = U(\cdots)U^\dagger. \quad (28)$$

We first note that terms of the form  $-i \text{ad}_{|j\rangle\langle j|}$  commute with the dissipative part  $\mathcal{D}_{|1\rangle\langle 2|}$  and according to [25] we can generate every element in  $-i \text{ad}_{u(d)}$  with

$u(d)$  the Lie algebra of  $d \times d$  hermitian matrices. Using  $\text{Ad}_{U(d)} = \exp(-i \text{ad}_u(d))$  with  $U(d)$  being the unitary group, we can get

$$\text{Ad}_U \circ \mathcal{D}_{|1\rangle\langle 2|} \circ \text{Ad}_{U^\dagger} = \mathcal{D}_{U|1\rangle\langle 2|U^\dagger} \quad (29)$$

for any  $U \in U(d)$ . Now we consider unitaries  $U$  that act as  $U|j\rangle = \sum_{k \in \mathcal{I}} c_k^{(j)} |k\rangle$  for  $j = 1, 2$  and  $\mathcal{I} = \{1, 2, 3, 4\}$ . We numerically verified that, from  $\mathcal{D}_{U|1\rangle\langle 2|U^\dagger}$ , thus created together with  $-i \text{ad}_H$  for all Hamiltonians  $H = \sum_{i,j \in \mathcal{I}} h_{ij} |i\rangle\langle j|$  having support on  $\mathcal{I}$ , all the operators of the form

$$\begin{cases} \mathcal{D}_{|i\rangle\langle j|, |k\rangle\langle l|} + \mathcal{D}_{|k\rangle\langle l|, |i\rangle\langle j|}, \\ i(\mathcal{D}_{|i\rangle\langle j|, |k\rangle\langle l|} - \mathcal{D}_{|k\rangle\langle l|, |i\rangle\langle j|}), \end{cases} \quad i, j, k, l \in \mathcal{I} \quad (30)$$

can be generated. Doing the same for different quartets  $\mathcal{I} = \{i, j, k, l\}$ , we are able to provide linearly independent operators (30) for all  $i, j, k, l \in \{1, \dots, d\}$ . Since any Lindbladian can be written in the Kossakowski form as a linear combination of those operators, it means that every Lindbladian can be generated through iterated commutators and linear combinations of the pair of generators  $\{\mathcal{L}_0, \mathcal{K}\}$  in (22) and (23). Given that this specific pair of Lindbladians is accessible, it then follows from the standard argument (see, e.g., [9]) that almost all pairs are. This was shown previously in a more abstract way by Kurniawan [7].

Now that we have found a pair  $\mathbf{L} = \{\mathcal{L}_0, \mathcal{K}\}$  that describes an accessible quantum system in arbitrary dimensions, we can make them commutative using a two-dimensional ( $d_A = 2$ ) auxiliary Hilbert space, i.e., we can purify them to

$$\tilde{\mathcal{L}}_0 = -2i \text{ad}_{H_0 \otimes |2\rangle\langle 2|} + \sqrt{2} \mathcal{D}_{|1\rangle\langle 2| \otimes |2\rangle\langle 2|}, \quad (31)$$

$$\tilde{\mathcal{K}} = -2i \text{ad}_{|1\rangle\langle 1| \otimes |1\rangle\langle 1|}. \quad (32)$$

Obviously on the extended Hilbert space the Lie algebra associated with the set  $\tilde{\mathbf{L}} = \{\tilde{\mathcal{L}}_0, \tilde{\mathcal{K}}\}$  is just two-dimensional,  $\dim \mathfrak{L}(\tilde{\mathbf{L}}) = 2$ , and the system is not accessible. If we perform frequent non-selective projective measurements on the auxiliary system described by the superprojector (11) with  $P_\pm = \mathbb{I} \otimes |\pm\rangle\langle \pm|$ , where  $|\pm\rangle$  are defined at the end of Sec. II, the original dynamics is recovered as (15) and the system becomes accessible. The existence of such a specific setup allows us to conclude [9] that *almost all open quantum systems become accessible by Zeno measurements.*

#### IV. CONCLUSIONS

We have generalized the work [12] on Hamiltonian purification by establishing a new and simple purification scheme for Lindbladians, which is also applicable to Hamiltonians. Given  $n$  Lindbladians, they can be made commutative by adding an  $n$ -dimensional auxiliary system to extend the Lindblad operators with hermitian projectors that form an orthonormal basis for the auxiliary

space. Through the projection by Zeno measurements for semigroup dynamics the original possibly noncommutative dynamics can be recovered by frequently measuring the auxiliary system in a non-selective way. The purification of more general dynamical maps with non-Markovian dynamics is left for future studies.

Moreover, we have proven that the pair of Lindbladians (22) and (23) describes an accessible open quantum system for generic system sizes, which tells us that generally a nonaccessible open quantum system is turned into an accessible one by frequent non-selective measurements. The model has also potential applications in simulating an arbitrary Markovian open system dynamics [26, 27] by steering it through control fields.

Clearly, the presented purification scheme also works for observables and density operators, although, except for the partial trace, an operational way that allows us to recover the original observables and states is not known to us. Since the noncommutativity is a unique feature of quantum mechanics, and in fact it was argued in [28, 29] that the noncommutativity distinguishes between quantum and classical mechanics, it is tempting to say that every quantum system can be made classical by purifying it to a larger space. However, we remark here that this is only the case in a dynamical sense, i.e. the dynamics can be made commutative but the observable algebra associated with the system still remains noncommutative.

#### ACKNOWLEDGMENTS

We like to thank John Gough for fruitful discussions. This work was supported by the Top Global University Project from the Ministry of Education, Culture, Sports, Science and Technology (MEXT), Japan. DB acknowledges support from EPSRC grant EP/M01634X/1. KY was supported by the Grant-in-Aid for Scientific Research (C) (No. 26400406) from the Japan Society for the Promotion of Science (JSPS) and by the Waseda University Grant for Special Research Projects (No. 2016K-215). HN was supported by the Waseda University Grant for Special Research Project (No. 2016B-173).

#### Appendix A: Projected Lindbladians

As an example of the fact that the projected counterpart  $\mathcal{P} \circ \mathcal{L} \circ \mathcal{P}$  of a Lindbladian  $\mathcal{L}$  does not generate proper quantum dynamics, consider for instance the case where  $\mathcal{L}$  describes a qubit amplitude damping with fixed point  $|0\rangle\langle 0|$  (this is characterized by a null Hamiltonian term  $H = 0$  and a unique Lindblad operator  $L_\alpha = |0\rangle\langle 1|$ ) and where the transformation  $\mathcal{P}$  is the dephasing map [13] associated with the canonical qubit base, i.e.,

$$\mathcal{P}(\cdots) = |0\rangle\langle 0|(\cdots)|0\rangle\langle 0| + |1\rangle\langle 1|(\cdots)|1\rangle\langle 1|. \quad (\text{A1})$$

Accordingly for an arbitrary density matrix  $\rho$  we have

$$\lim_{t \rightarrow \infty} (e^{(\mathcal{P} \circ \mathcal{L} \circ \mathcal{P})t} \circ \mathcal{P})\rho = |0\rangle\langle 0|, \quad (\text{A2})$$

while on the contrary

$$\begin{aligned} \lim_{t \rightarrow \infty} e^{(\mathcal{P} \circ \mathcal{L} \circ \mathcal{P})t} \rho &= |0\rangle\langle 0| + (\text{id} - \mathcal{P})\rho \\ &= |0\rangle\langle 0| + |0\rangle\langle 0|\rho|1\rangle\langle 1| + |1\rangle\langle 1|\rho|0\rangle\langle 0|, \end{aligned} \quad (\text{A3})$$

which in general is not a valid state.

## Appendix B: Derivation of the Projected Dynamics

We start by noticing that given a generic non-selective transformation  $\mathcal{P}$  as in (11) and the unitary generator  $\mathcal{K}$  with Hamiltonian  $H$ , the following identity holds

$$(\mathcal{P} \circ \mathcal{K} \circ \mathcal{P})(\dots) = -i \sum_k [H^{(k)}, P_k(\dots)P_k], \quad (\text{B1})$$

where  $H^{(k)} = P_k H P_k$ . Similarly, given a dissipator  $\mathcal{D}$  characterized by Lindblad operators  $L_\alpha$  we have

$$\begin{aligned} (\mathcal{P} \circ \mathcal{D} \circ \mathcal{P})(\dots) &= \sum_{\alpha, k, k'} \left[ 2L_\alpha^{(kk')}(\dots)L_\alpha^{(kk')\dagger} - L_\alpha^{(kk')\dagger}L_\alpha^{(kk')}(\dots)P_{k'} \right. \\ &\quad \left. - P_{k'}(\dots)L_\alpha^{(kk')\dagger}L_\alpha^{(kk')} \right], \end{aligned} \quad (\text{B2})$$

with  $L_\alpha^{(kk')} = P_k L_\alpha P_{k'}$ . Assume next  $H$  and  $L_\alpha$  as those associated with the Lindbladian  $\tilde{\mathcal{L}}_j$  with (10), and  $P_k$  as in (14). Since  $\{|\phi_k\rangle\}$  is mutually unbiased with respect to  $\{|j\rangle\}$  the following identity holds,

$$\langle j|\phi_k\rangle = e^{-i\varphi_{jk}}/\sqrt{d_A}, \quad (\text{B3})$$

with  $\varphi_{jk}$  generic phases, and hence

$$\tilde{H}_j^{(k)} = P_k \tilde{H}_j P_k = H_j \otimes |\phi_k\rangle\langle\phi_k| = H_j P_k, \quad (\text{B4})$$

$$\tilde{L}_{j,\alpha}^{(kk')} = P_k \tilde{L}_{j,\alpha} P_{k'} = e^{i(\varphi_{jk} - \varphi_{jk'})} L_{j,\alpha} \otimes |\phi_k\rangle\langle\phi_{k'}|/\sqrt{d_A}, \quad (\text{B5})$$

$$\tilde{L}_{j,\alpha}^{(kk')\dagger} \tilde{L}_{j,\alpha}^{(kk')} = L_{j,\alpha}^\dagger L_{j,\alpha} \otimes |\phi_{k'}\rangle\langle\phi_k| = L_{j,\alpha}^\dagger L_{j,\alpha} P_{k'}/d_A. \quad (\text{B6})$$

Inserting these into (B1) and (B2) we then obtain

$$(\mathcal{P} \circ \tilde{\mathcal{K}}_j \circ \mathcal{P})(\dots) = (\mathcal{K}_j \circ \mathcal{P})(\dots), \quad (\text{B7})$$

$$\begin{aligned} (\mathcal{P} \circ \tilde{\mathcal{D}}_j \circ \mathcal{P})(\dots) &= (\mathcal{D}_j \circ \mathcal{P})(\dots) \\ &\quad + 2 \sum_\alpha L_{j,\alpha} \mathcal{T}(\dots) L_{j,\alpha}^\dagger, \end{aligned} \quad (\text{B8})$$

that is

$$(\mathcal{P} \circ \tilde{\mathcal{L}}_j \circ \mathcal{P})(\dots) = (\mathcal{L}_j \circ \mathcal{P})(\dots) + 2 \sum_\alpha L_{j,\alpha} \mathcal{T}(\dots) L_{j,\alpha}^\dagger, \quad (\text{B9})$$

where  $\mathcal{T}$  is the super-operator

$$\mathcal{T}(\dots) = \frac{1}{d_A} \mathbb{1}_A \text{Tr}_A[(\dots)] - \mathcal{P}(\dots), \quad (\text{B10})$$

with  $\text{Tr}_A[(\dots)]$  indicating the partial trace over the auxiliary system,  $A$  and  $\mathbb{1}_A$  being the identity operator on the associated Hilbert space.

In order to prove (15) let us now focus on the evolution induced by CPTP map  $\Phi_{t,\infty}^{(\mathcal{L}_j)}$  in (13) associated with the  $j$ th element of  $\mathbf{L}$  on a generic density matrix  $\rho(0)$  of the joint system  $Q + A$ , i.e.,  $\rho(t) = \Phi_{t,\infty}^{(\mathcal{L}_j)} \rho(0)$ . We are interested in the dynamics of the reduced density matrix of  $Q$ , i.e.,

$$\rho_Q(t) = \text{Tr}_A[\rho(t)] = \text{Tr}_A[\Phi_{t,\infty}^{(\mathcal{L}_j)} \rho(0)]. \quad (\text{B11})$$

By taking the first derivative with respect to  $t$  and using (B9) we obtain

$$\begin{aligned} \frac{\partial}{\partial t} \rho_Q(t) &= \text{Tr}_A[(\mathcal{P} \circ \tilde{\mathcal{L}}_j \circ \mathcal{P})\rho(t)] \\ &= \mathcal{L}_j(\text{Tr}_A[\mathcal{P}\rho(t)]) + 2 \sum_\alpha L_{j,\alpha} \text{Tr}_A[\mathcal{T}\rho(t)] L_{j,\alpha}^\dagger, \end{aligned} \quad (\text{B12})$$

which finally yields the thesis

$$\frac{\partial}{\partial t} \rho_Q(t) = \mathcal{L}_j \rho_Q(t) \implies \rho_Q(t) = e^{\mathcal{L}_j t} \rho_Q(0), \quad (\text{B13})$$

by noticing that

$$\text{Tr}_A[\mathcal{P}\rho(t)] = \rho_Q(t), \quad \text{Tr}_A[\mathcal{T}\rho(t)] = 0. \quad (\text{B14})$$

## Appendix C: An Accessible Pair of Lindbladians

Here we show that the pair of Lindbladians  $\{\mathcal{L}_0, \mathcal{K}\}$  in (22) and (23) generates an accessible system. We use the notations (25)–(28). First of all, we show that  $\mathcal{K} = -i \text{ad}_{|j\rangle\langle j|}$  commutes with the dissipative part  $\mathcal{D}_{|1\rangle\langle 2|}$  of  $\mathcal{L}_0$  in (22). Using an identity [30]

$$[-i \text{ad}_H, \mathcal{D}_A] = \frac{1}{2} \mathcal{D}_{A-i[H,A]} - \frac{1}{2} \mathcal{D}_{A+i[H,A]}, \quad (\text{C1})$$

we have

$$\begin{aligned} [-i \text{ad}_{|j\rangle\langle j|}, \mathcal{D}_{|1\rangle\langle 2|}] &= \frac{1}{2} \mathcal{D}_{|1\rangle\langle 2|-i|1\rangle\langle 2|\delta_{j1}+i|1\rangle\langle 2|\delta_{j2}} \\ &\quad - \frac{1}{2} \mathcal{D}_{|1\rangle\langle 2|+i|1\rangle\langle 2|\delta_{j1}-i|1\rangle\langle 2|\delta_{j2}}. \end{aligned} \quad (\text{C2})$$

For  $j \neq 1, 2$  it trivially vanishes, while for  $j = 1, 2$  we get  $\frac{1}{2}(|1 \mp i|^2 - |1 \pm i|^2) \mathcal{D}_{|1\rangle\langle 2|} = 0$ , where we have used  $\mathcal{D}_{\alpha A} = |\alpha|^2 \mathcal{D}_A$ . This commutativity implies that we can generate every  $-i \text{ad}_{u(d)}$  (see [25]) and thus every  $\text{Ad}_U \circ \mathcal{D}_{|1\rangle\langle 2|} \circ \text{Ad}_{U^\dagger} = \mathcal{D}_{U|1\rangle\langle 2|U^\dagger}$  for any  $U \in U(d)$ .



Taking unitaries  $U|j\rangle = \sum_{k \in \mathcal{I}} c_k^{(j)}|k\rangle$  for  $j = 1, 2$  and  $\mathcal{I} = \{1, 2, 3, 4\}$ , we have

$$\mathcal{D}_{U|1\rangle\langle 2|U^\dagger} = \sum_{i,j,k,l \in \mathcal{I}} (c_i^{(1)})^* c_j^{(2)} c_k^{(1)} (c_l^{(2)})^* \mathcal{D}_{|i\rangle\langle j|, |k\rangle\langle l|}. \quad (\text{C3})$$

We numerically verified that all the operators of the form (30) for  $\mathcal{I} = \{1, 2, 3, 4\}$  can be obtained by linear combinations of  $\mathcal{D}_{U|1\rangle\langle 2|U^\dagger}$  and  $-i \text{ad}_H$  with different  $U$  and

$H = \sum_{i,j \in \mathcal{I}} h_{ij}|i\rangle\langle j|$  on  $\mathcal{I} = \{1, 2, 3, 4\}$ . The same argument applies to any quartets  $\mathcal{I} = \{i, j, k, l\}$ , and all the operators of the form (30) for all  $\mathcal{I}$  are available. Then, every Lindbladian can be given as a linear combination of those operators, i.e.,

$$\mathcal{L} = \sum_{i,j,k,l} [c_{ijkl}^+ (\mathcal{D}_{|i\rangle\langle j|, |k\rangle\langle l|} + \mathcal{D}_{|k\rangle\langle l|, |i\rangle\langle j|}) + c_{ijkl}^- i(\mathcal{D}_{|i\rangle\langle j|, |k\rangle\langle l|} - \mathcal{D}_{|k\rangle\langle l|, |i\rangle\langle j|})] \quad (\text{C4})$$

with some coefficients  $c_{ijkl}^\pm$ .

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